UNIT-II

VECTOR CALCULUS

Directional derivative

The derivative of a point function (scalar or vector) in a particular direction is called its directional derivative along the direction.

The directional derivative of a scalar point function ϕ in a given direction is the rate of change of ϕ in the direction. It is given by the component of grad ϕ in that direction.

The directional derivative of a scalar point function ϕ (x,y,z) in the direction of $\vec{a}$ is given by $\frac{\nabla\phi.\vec{a}}{|\vec{a}|}$.

Directional derivative of ϕ is maximum in the direction of $\nabla\phi$. Hence the maximum directional derivative is $|\nabla\phi| or |grad\phi|$

Unit normal vector to the surface

If ϕ (x, y, z) be a scalar function, then ϕ (x, y, z) = c represents A surface and the unit normal vector to the surface ϕ is given by

$$\frac{\nabla\phi}{|\nabla\phi|}$$

Equation of the tangent plane and normal to the surface

Suppose $\vec{a}$ is the position vector of the point (x_0, y_0, z_0) On the surface ϕ (x, y, z) = c. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of any point (x,y,z) on the tangent plane to the surface at $\vec{a}$, then the equation of the tangent plane to the surface ϕ at a given point $\vec{a}$ on it is given by $\left(\vec{r} - \vec{a}\right).grad\phi = 0$

If $\vec{r}$ is the position vector of any point on the normal to the surface at the point $\vec{a}$ on it. The vector equation of the normal at a given point $\vec{a}$ on the surface ϕ is $\left(\vec{r} - \vec{a}\right)\times grad\phi = 0$

The Cartesian form of the normal at (x_0, y_0, z_0) on the surface ϕ (x,y,z) = c is

$$\frac{x - x_0}{\frac{\partial\phi}{\partial x}} = \frac{y - y_o}{\frac{\partial\phi}{\partial y}} = \frac{z - z_0}{\frac{\partial\phi}{\partial z}}$$

Divergence of a vector

If $\vec{F}(x, y, z)$ is a continuously differentiable vector point function in a given region of space, then the divergences of $\vec{F}$ is defined by

$$\nabla.\vec{F} = div\vec{F} = \vec{i}\frac{\partial\vec{F}}{\partial x} + \vec{j}\frac{\partial\vec{F}}{\partial y} + \vec{k}\frac{\partial\vec{F}}{\partial z}$$

$$=\sum \vec{i}\,\frac{\partial \vec{F}}{\partial x}$$

If $\vec{F}=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$ **,then** $div\,\vec{F}=\nabla.(F_1\vec{i}+F_2\vec{j}+F_3\vec{k})$

i.e., $div\,\vec{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$

Solenoidal Vector

A vector $\vec{F}$ **is said to be solenoidal if** $div\,\vec{F}=0$ **(ie)** $\nabla.\vec{F}=0$

Curl of vector function

If $\vec{F}(x,y,z)$ **is a differentiable vector point function defined at each point (x, y, z), then the curl of** $\vec{F}$ **is defined by**

$$curl\,\vec{F}=\nabla\times\vec{F}$$

$$=\vec{i}\times\frac{\partial \vec{F}}{\partial x}+\vec{j}\times\frac{\partial \vec{F}}{\partial y}+\vec{k}\times\frac{\partial \vec{F}}{\partial z}$$

$$=\sum \vec{i}\times\frac{\partial \vec{F}}{\partial x}$$

If $\vec{F}=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$ **,then** $curl\,\vec{F}=\nabla\times(F_1\vec{i}+F_2\vec{j}+F_3\vec{k})$

$$curl\,\vec{F}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$=\vec{i}\left[\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right]-\vec{j}\left[\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right]+\vec{k}\left[\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right]$$

Curl $\vec{F}$ **is also said to be rotation** $\vec{F}$

Irrotational Vector

A vector $\vec{F}$ **is called irrotational if Curl** $\vec{F}=0$

(ie) if $\nabla\times\vec{F}=0$

Scalar Potential

If $\vec{F}$ **is an irrotational vector, then there exists a scalar function** ϕ

Such that $\vec{F}=\nabla\phi$**. Such a scalar function is called scalar potential of** $\vec{F}$

Properties of Gradient

1. **If f and g are two scalar point function that** $\nabla(f\pm g)=\nabla f\pm\nabla g$ **(or)** $grad(f\pm g)=gradf\pm gradg$

Solution: $\nabla(f\pm g)=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(f\pm g)$

$$= \left(\vec{i}\frac{\partial}{\partial x}(f \pm g) + \vec{j}\frac{\partial}{\partial y}(f \pm g) + \vec{k}\frac{\partial}{\partial z}(f \pm g) \right)$$

$$= \vec{i}\frac{\partial f}{\partial x} \pm \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial f}{\partial y} \pm \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial f}{\partial z} \pm \vec{k}\frac{\partial g}{\partial z}$$

$$= \left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} \right) \pm \left(\vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z} \right)$$

$$= \nabla f \pm \nabla g$$

2. If f and g are two scalar point functions then $\nabla(fg) = f\nabla g + g\nabla f$ **(or)** $grad(fg) = fgradg + ggradf$

Solution: $$\nabla(fg) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right)(fg)$$

$$= \left(\vec{i}\frac{\partial}{\partial x}(fg) + \vec{j}\frac{\partial}{\partial y}(fg) + \vec{k}\frac{\partial}{\partial z}(fg) \right)$$

$$= \vec{i}\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x} \right) + \vec{j}\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y} \right) + \vec{k}\left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z} \right)$$

$$= f\left(\vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z} \right) + g\left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} \right)$$

$$= f\nabla g + g\nabla f$$

3. If f and g are two scalar point function then $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ **where** $g \neq 0$

Solution: $$\nabla\left(\frac{f}{g}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right)\left(\frac{f}{g}\right)$$

$$= \sum \vec{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right)$$

$$= \sum \vec{i}\left(\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2} \right)$$

$$= \frac{1}{g^2}\left(g\sum \vec{i}\frac{\partial f}{\partial x} - f\sum \vec{i}\frac{\partial g}{\partial x} \right)$$

$$= \frac{1}{g^2}\left[g\nabla f - f\nabla g \right]$$

4. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ **such that** $\left|\vec{r}\right| = r$ **,prove that** $\nabla r^n = nr^{n-2}\vec{r}$

Solution: $$\nabla r^n = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right)r^n$$

$$= \left(\vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z} \right)$$

$$= \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= \frac{nr^{n-1}}{r}\left[x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right]$$

$$= \frac{nr^{n-1}}{r}\vec{r}$$

5. Find a unit normal to the surface $x^2y + 2xz = 4$ **at (2,-2, 3)**

Solution: Given that $\phi = x^2y + 2xz$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2y + 2xz)$$

$$= \vec{i}(2xy + 2z) + \vec{j}(x^2) + \vec{k}(2x)$$

At (2,-2, 3)

$$\nabla\phi = \vec{i}(-8+6) + \vec{j}(4) + \vec{k}(4)$$

$$= -2\,\vec{i} + 4\,\vec{j} + 4\,\vec{k}$$

$$|\nabla\phi| = \sqrt{4+16+16} = \sqrt{36} = 6$$

Unit normal to the given surface at (2,-2,3)

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\,\vec{i} + 4\,\vec{j} + 4\,\vec{k}}{6}$$

$$= \frac{1}{3}\left(-\vec{i} + 2\,\vec{j} + 2\,\vec{k}\right)$$

6. Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ **at (1,2,3) in the direction of** $2\,\vec{i} + \vec{j} - \vec{k}$

Solution: Given $\phi = x^2yz + 4xz^2 + xyz$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2 + xyz)$$

$$= (2xyz + 4z^2 + yz)\vec{i} + (x^2z + xz)\vec{j} + (x^2y + 8xz + xy)\vec{k}$$

At (1, 2, 3)

$$\nabla\phi = 54\,\vec{i} + 6\,\vec{j} + 28\,\vec{k}$$

Given: $\vec{a} = 2\,\vec{i} + \vec{j} - \vec{k}$

$$\therefore |\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$$

$$\therefore D.D = \nabla\phi.\frac{\vec{a}}{|\vec{a}|}$$

$$= \left(54\vec{i}+6\vec{j}+28\vec{k}\right).\frac{2\vec{i}+2\vec{j}-\vec{k}}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}[108+6-28] = \frac{1}{\sqrt{6}}[86]$$

7. Find the angle between the surface $x^2+y^2+z^2=5$ and $x^2+y^2+z^2-2x=5$ at (0,1,2)

Solution: Let $\phi_1 = x^2+y^2+z^2$ **and** $\phi_2 = x^2+y^2+z^2-2x$

$$\frac{\partial\phi_1}{\partial x} = 2x, \frac{\partial\phi_1}{\partial y} = 2y, \frac{\partial\phi_1}{\partial z} = 2z$$

$$\frac{\partial\phi_2}{\partial x} = 2x-2, \frac{\partial\phi_2}{\partial y} = 2y, \frac{\partial\phi_2}{\partial z} = 2z$$

$$\nabla\phi_1 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$$

$$\nabla\phi_2 = (2x-2)\vec{i}+2y\vec{j}+2z\vec{k}$$

At (o,1,2)

$$\nabla\phi_1 = 2\vec{j}+4\vec{k}$$

$$\nabla\phi_2 = -2\vec{i}+2\vec{j}+4\vec{k}$$

Cos $\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{\left(2\vec{j}+4\vec{k}\right).\left(-2\vec{i}+2\vec{j}+4\vec{k}\right)}{\sqrt{16+4}\sqrt{4+4+6}}$

$$\cos\theta = \frac{4+16}{\sqrt{20}\sqrt{24}} = \frac{20}{\sqrt{20}\sqrt{24}}$$

$$\theta = \cos^{-1}\left(\frac{20}{\sqrt{20}\sqrt{24}}\right)$$

$$= \cos^{-1}\left(\frac{\sqrt{20}}{\sqrt{24}}\right)$$

8. Find the angle between the surfaces $x\log z = y^2-1$ and $x^2y = 2-z$ at the point (1,1,1)

Solution: let $\phi_1 = y^2 - x\log z$ **and** $\phi_2 = x^2y+z$

$$\frac{\partial\phi_1}{\partial x} = -\log z, \frac{\partial\phi_1}{\partial y} = 2y, \frac{\partial\phi_1}{\partial z} = -\frac{x}{z}$$

$$\frac{\partial\phi_2}{\partial x} = 2xy, \frac{\partial\phi_2}{\partial y} = x^2, \frac{\partial\phi_2}{\partial z} = 1$$

$$\nabla\phi_1 = (-\log z)\vec{i}+2y\vec{j}-\frac{k}{z}\vec{k}$$

$$\nabla\phi_2 = 2\vec{j}-\vec{k}$$

$$\textbf{Cos}\,\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{2-1}{\sqrt{4+1}\sqrt{4+1+1}} = \frac{1}{\sqrt{5}\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}\sqrt{6}}\right)$$

9. Find $\nabla^2(r^n)$

Solution: $\nabla^2(r^n) = \nabla.\nabla(r^n)$

$$= \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}\,nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\,nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\left|\vec{r}\right| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \nabla^2(r^n) = nr^{n-1}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= nr^{n-2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= nr^{n-2}\left[\vec{r}\right]$$

Since $\nabla\left(\phi\vec{u}\right) = \nabla\phi.\vec{u} + \phi\,div\,\vec{u}$

$$\nabla^2(r^n) = \nabla\left(nr^{n-2}\vec{r}\right)$$

$$= nr^{n-2}\left(\nabla.\vec{r}\right) + \nabla\left(nr^{n-2}\right).\vec{r}$$

$$\nabla.\vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

=1+1+1 = 3

$$\nabla^2(r^n) = 3nr^{n-2} + n\nabla\left(r^{n-2}\right).\vec{r}$$

$$= 3nr^{n-2} + n(n-2)\left(r^{n-4}\right).r^2$$

$$= 3nr^{n-2} + n(n-2)\left(r^{n-2}\right)$$

$$\nabla^2(r^n) = r^{n-2}\left[n^2 + n\right] = n(n+1)r^{n-2}$$

10. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $|\vec{r}| = r$ .Prove that $r^n\vec{r}$ is solenoidal if $n = -3$ and $r^n\vec{r}$ is irrotational for all vectors of n.

Solution: $r^n\vec{r} = r^n x\vec{i} + r^n y\vec{j} + r^n\vec{k}$

$$\textbf{div}\left(r^n\vec{r}\right) = \frac{\partial}{\partial x}\left(r^n x\right) + \frac{\partial}{\partial y}\left(r^n y\right) + \frac{\partial}{\partial z}\left(r^n z\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now $r^2 = x^2 + y^2 + z^2$

Differentiating partially w.r.to x,

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly,
$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$
$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\frac{\partial}{\partial x}\left(r^n x\right) = x\frac{\partial}{\partial r}\left(r^n\right).\frac{\partial r}{\partial x} + r^n$

$$= \textbf{x.n}\, r^{n-1}\frac{x}{r} + r^n$$

$$\frac{\partial}{\partial y}\left(r^n y\right) = nr^{n-2}y^2 + r^n$$

$$\frac{\partial}{\partial z}\left(r^n z\right) = nr^{n-2}z^2 + r^n$$

From (1) we have

$$div\left(r^n\vec{r}\right) = nr^{n-2}\left(x^2 + y^2 + z^2\right) + 3r^n$$
$$= nr^n + 3r^n$$
$$= (n+3)r^n$$

The vector $r^n\vec{r}$ **is solenoidal if** $div\left(r^n\vec{r}\right)$ **= 0**

$$\Rightarrow (n+3)r^n = 0$$
$$\Rightarrow n + 3 = 0$$
$$\Rightarrow n = -3$$

$\therefore r^n\vec{r}$ **is solenoidal only if n = -3**

Now $curl\left(r^n\vec{r}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$

$$= \sum\vec{i}\left(\frac{\partial}{\partial y}\left(r^n z\right) - \frac{\partial}{\partial z}\left(r^n y\right)\right)$$

$$= \sum\vec{i}\left(nr^{n-1}\frac{\partial r}{\partial y}z - nr^{n-1}y\frac{\partial r}{\partial z}\right)$$

$$= \sum \vec{i}\left(nr^{n-1}\frac{y}{r}z - nr^{n-1}y\frac{z}{r}\right)$$

$$= \sum \vec{i}\left(nr^{n-2}yz - nr^{n-2}yz\right)$$

$$= \mathbf{0}$$

Curl ($r^n \vec{r}$ **) =** $0\vec{i} + 0\vec{j} + 0\vec{k}$ **=0**

Curl ($r^n \vec{r}$ **) = 0 for all values of n**

Hence $r^n \vec{r}$ **is irrotational for all values of n.**

11. Prove that $\vec{F} = \left(y^2\cos x + z^3\right)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ **is irrotational and find its scalar potential**

Solution:

$$curl\left(\vec{F}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}\left[3z^2 - 3z^2\right] + \vec{k}[2y\cos x - 2y\cos x] = 0$$

$\therefore \vec{F}$ **is irrotational.**

To Find ϕ **such that** $\vec{F} = grad\phi$

$$\therefore \left(y^2\cos x + z^2\right)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Integrating the equation partially w.r.to x,y,z respectively

$$\phi = y^2\sin x + xz^3 + f_1(y,z)$$

$$\phi = y^2\sin x - 4y + f_2(x,z)$$

$$\phi = xz^3 + f_3(x,y)$$

$\therefore \phi = y^2\sin x + xz^3 - 4y + C,$ **is scalar potential**

12. Prove that $div\left(\vec{A}\times\vec{B}\right) = \vec{B}.(curl\,\vec{A}) - \vec{A}.(curl\,\vec{B})$

Proof : $div\left(\vec{A}\times\vec{B}\right) = \nabla.(\vec{A}\times\vec{B})$

$$= \sum \vec{i}\frac{\partial}{\partial x}\left(\vec{A}\times\vec{B}\right)$$

$$= \sum \vec{i}\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right) + \sum \vec{i}\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)$$

$$= -\sum \vec{i}\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right) + \sum \vec{i}\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)$$

$$= -\left(\sum \vec{i} \times \frac{\partial \vec{B}}{\partial x}\right).\vec{A} + \left(\sum \vec{i} \times \frac{\partial \vec{A}}{\partial x}\right).\vec{B}$$

$$= -\left(curl\,\vec{B}\right).\vec{A} + \left(curl\,\vec{A}\right).\vec{B}$$

13.Prove that $curl\left(curl\,\vec{F}\right) = \nabla\left(\nabla \bullet \vec{F}\right) - \nabla^2 \vec{F}$

Solution:

$$curl\left(curl\,\vec{F}\right) = \nabla \times \left(\nabla \times \vec{F}\right)$$

By using $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a}.\vec{c}\right)\vec{b} - \left(\vec{a}.\vec{b}\right)\vec{c}$

$$= \left(\nabla.\vec{F}\right)\nabla - (\nabla.\nabla)\vec{F}$$

$$= \left(\nabla.\vec{F}\right)\nabla - \nabla^2 \vec{F}$$

VECTOR INTEGRATION
Line, surface and Volume Integrals

Problems based on line Integral
Example 1:

If $\vec{F} = \left(3x^2 + 6y\right)\vec{i} - 14yz\,\vec{j} + 20xz^2\,\vec{k}$ **Evaluate** $\int_C \vec{F}.dr$ **from (0,0,0) to (1,1,1) along the curve** $x = t, y = t^2, z = t^3$

Solution: The end points are (0, 0, 0) and (1, 1, 1)
These points correspond to t = 0 and t = 1
$\therefore dx = dt, dy = 2t, dz = 3t^2$

$$\int_C \vec{F}.dr = \int_C \left(3x^2 + 6y\right)dx - 14yzdy + 20xz^2dz$$

$$= \int_0^1 \left(3t^2 + 6t^2\right)dt - 14t^5(2tdt) + 20t^7\left(3t^2\right)dt$$

$$= \int_0^1 \left(9t^2 - 28t^6 + 60t^9\right)dt$$

$$= \left(3t^3 - 4t^7 + 6t^{10}\right)_0^1$$

$$= [(3 - 4 + 6) - 0] = 5$$

Example 2:

Show that $\vec{F} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$ **is a conservative vector field.**

Solution: If $\vec{F}$ is conservative then $\nabla \times \vec{F} = 0$

$$\textbf{Now}\quad \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = 0$$

$\therefore \vec{F}$ **is a conservative vector field.**

Surface Integrals

Definition: Consider a surface S. Let n denote the unit outward normal to the surface S. Let R be the projection of the surface x on the XY plane. Let $\vec{f}$ be a vector valued defined in some region containing the surface S. Then the surface integral of $\vec{f}$ is defined to be $\iint_S \vec{f}.\hat{n}ds = \iint_R \frac{\vec{f}.\hat{n}}{\left|\vec{n}.\vec{k}\right|}dx.dy$

Example 1;

Evaluate $\iint_S \vec{f}.\hat{n}ds$ **where** $\vec{F} = z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}$ **and S is the surface of the cylinder** $x^2 + y^2 = 1$ **included in the first octant between the planes z = 0 and z = 2.**

Solution: Given $\vec{F} = z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}$

$$\phi = x^2 + y^2 - 1$$

$$\nabla\phi = 2x\,\vec{i} + 2y\,\vec{j}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2}$$

$$= 2\sqrt{x^2 + y^2}$$

$$= 2$$

The unit normal $\hat{n}$ to the surface = $\frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2xi + 2yj}{2} = xi + yj$$

$$\vec{F}.\hat{n} = \left(z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}\right).\left(x\,\vec{i} + y\,\vec{j}\right) = xz + xy$$

$$\text{Now} \iint_S \vec{F}.\hat{n}\,dS = \iint_R \vec{F}.\hat{n}\frac{dy\,dz}{|\hat{n}.\vec{i}|}$$

Where R is the projection of S on YZ plane.

$$= \iint_R (xz + xy)\frac{dy\,dz}{x} \qquad [\because \hat{n}\cdot\hat{i} = (x\hat{i} + y\hat{j})\cdot\hat{i} = x]$$

$$= \iint_R (z + y)\,dy\,dz$$

$$= \int_0^2 \int_0^1 (z + y)\,dy\,dz$$

$$= \int_0^2 \left[zy + \frac{y^2}{2}\right]_0^1 dz = \int_0^2 \left(z + \frac{1}{2}\right) dz$$

$$= \left[\frac{z^2}{2} + \frac{1}{2}z\right]^2$$

$$= \left(\frac{4}{2} + \frac{2}{2}\right) - (0 + 0)$$

$$= (2 + 1) = 3.$$

Example **Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$ where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ as S is the part of the plane $2x + 3y + 6z = 12$ which is in the first octant.**

Solution : Given : $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$

Let $\phi = 2x + 3y + 6z - 12$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= 2\vec{i} + 3\vec{j} + 6\vec{k}$$

$$|\nabla\phi| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{7}$$

$$\vec{F}\cdot\hat{n} = (18z\vec{i} - 12\vec{j} + 3y\vec{k})\cdot\left(\frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{7}\right)$$

$$= \frac{36z - 36 + 18y}{7}$$

$$= \frac{1}{7}\left[36\left(\frac{12-2x-3y}{6}\right) - 36 + 18y\right] \qquad \left[\because z = \frac{12-2x-3y}{6}\right]$$

$$= \frac{1}{7}[6(12-2x-3y) - 36 + 18y] = \frac{1}{7}[36 - 12x]$$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \frac{36-12x}{7}\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$\hat{n}\cdot\vec{k} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}\cdot\vec{k} = \frac{6}{7}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \frac{36-12x}{7}\,\frac{7}{6}\,dx\,dy$$

$$= \iint_R (6-2x)\,dx\,dy$$

Given plane equation is $2x + 3y + 6z = 12$ in XOY area $z = 0$.

$$2x + 3y = 12$$

$$3y = 12 - 2x$$

$$y = \frac{12-2x}{3}$$

$\therefore$ y varies from O to $\frac{12-2x}{3}$ in X plane $y = 0$, $z = 0$

$$\therefore 2x = 12$$

$$x = 6$$

$\therefore$ x varies from 0 to 6

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} (6-2x)\,dy\,dx$$

$$= \int_0^6 (6y - 2xy)_0^{\frac{12-2x}{3}}\,dx$$

$$= \int_0^6 \left[6\left(\frac{12-2x}{3}\right) - 2x\left(\frac{12-2x}{3}\right)\right] dx$$

$$= \int_0^6 \left(24 - 4x - 8x + \frac{4x^2}{3}\right) dx$$

$$= \int_0^6 \left(24 - 12x + \frac{4x^2}{3}\right) dx$$

$$= \left[24x - 6x^2 + \frac{4x^3}{9}\right]_0^6 = 24.$$

VOLUME INTEGRALS

The volume integral of F (x, y, z) over a region enclosing a volume V is given by $\iiint_V$ F $(x,y,z)\, dV$ or $\iiint_V$ F $(x,y,z)\, dx\, dy\, dz$.

PROBLEMS BASED ON VOLUME INTEGRALS

Example 2.3.16. If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, evaluate $\iiint_V \nabla \times \vec{F}\, dV$ where V is the region bounded by x = 0, y = 0, z = 0 and 2x + 2y + z = 4.

Solution :
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}(0) + \vec{j}(-3 + 4) + \vec{k}(-2y - 0)$$

$$= \vec{j} - 2y\vec{k}$$

$$\therefore \iiint_V \nabla \times \vec{F}\, dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\vec{j} - 2y\vec{k})\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (z\vec{j} - 2yz\vec{k})_0^{4-2x-2y}\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} [4 - 2x - 2y]\, j - 2y\,(4 - 2x - 2y)\,\vec{k}]\, dy\, dx$$

$$= \int_0^2 \left[(4y - 2xy - y^2)\vec{j} - \left(4y^2 - 2xy^2 - \frac{4y^3}{3}\right) k\right]_0^{2-x} dx$$

$$= \int_0^2 \left[\left[4(2-x) - 2x(2-x) - (2-x)^2 \right] j - \left[4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3 \right] k \right] dx$$

$$= \int_0^2 (8 - 4x - 4x + 2x^2 - 4 - x^2 + 4x)\vec{j} - [(16 - 16x + 4x^2 - 8x + 8x^2 - 2x^3) - \frac{4}{3}(8 - 12x + 6x^2 - x^3) k] dx$$

$$= \int_0^2 \left[(4 - 4x + x^2)\vec{j} - \frac{\vec{k}}{3}(16 - 24x + 12x^2 - 2x^3 \right] dx$$

$$= \left[4x - 2x^2 + \frac{x^3}{3} \right] \vec{j} - \frac{\vec{k}}{3} \left[16x - 12x^2 + 4x^3 - \frac{x^4}{2} \right]_0^2$$

$$= \left(8 - 8 + \frac{8}{3} \right) \vec{j} - \frac{\vec{k}}{3}(32 - 48 + 32 - 8) = \frac{8}{3}(\vec{j} - \vec{k})$$

INTEGRAL THEOREMS

(i) Gauss's divergence theorem
(ii) Stoke's theorem
(iii) Green's theorem in the plane

Green's Theorem
Statement:

If M(x,y) and N(x,y) are continuous functions with continuous partial derivatives in a region R of the xy plane bounded by a simple closed curve C, then

$\oint_c Mdx + ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$**, where C is the curve described in the positive direction.**

Verify Green's theorem in a plane for the integral $\int_c (x-2y)dx + xdy$

taken around the circle $x^2 + y^2 = 4$

Solution: Green's theorem gives

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Consider $\int_c (x-2y)dx + xdy$

M = x – 2y N = x

$$\frac{\partial M}{\partial y} = -2, \frac{\partial N}{\partial x} = 1$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$\iint_R (1+2)dxdy = 3\iint_R dxdy$$

= 3[Area of the circle]

= **3** πr^2

=**3.** π.4

=12π **……………………(1)**

Now $\int Mdx + Ndy$

We know that the parametric equation of the circle $x^2 + y^2 = 4$

x = 2 cos θ **y = 2 sin** θ

$dx = -2\sin\theta d\theta$ **,** $dy = 2\cos\theta d\theta$

$\therefore Mdx + Ndy = (x - 2x)dx + xdy$

$= (2\cos\theta - 4\sin\theta)(-2\sin\theta d\theta) + 2\cos\theta(2\cos\theta)d\theta$

$= -2\cos\theta\sin\theta + 8\sin^2\theta + 4\cos^2\theta d\theta$

Where θ **various from 0 to** 2π

$$\therefore \int_C Mdx + Ndy = \int_0^{2\pi} (-2\cos\theta\sin\theta + 4\sin^2\theta + 4)d\theta$$

$$= \int_0^{2\pi} \left(-\sin 2\theta + 4\left(\frac{1-\cos 2\theta}{2} \right) + 4 \right) d\theta$$

$$= \int_0^{2\pi} (-\sin 2\theta + 6 - 2\cos 2\theta)d\theta$$

$$= \left[\frac{\cos 2\theta}{2} + 6\theta - \frac{2\sin 2\theta}{2} \right]_0^{2\pi}$$

$= \frac{1}{2} + 12\pi - \frac{1}{2} = 12\pi$ **……………………..(2)**

$\therefore$ **From (1) and (2)**

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Hence Green's Theorem is verified.

Example 2

Using Green's theorems find the area of a circle of radius r.

Solution: By Green's theorem we know that

Area enclosed by C = $\frac{1}{2}\int_C xdy - ydx$

The parametric equation of a circle of radius r is x = $r\cos\theta, y = r\sin\theta$

Where $0 \le \theta \le 2\pi$

$\therefore$ **Area of the circle =** $\frac{1}{2}\int_0^{2\pi} r\cos\theta(r\cos\theta) - r\sin\theta(-r\sin\theta)d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \left(r^2\cos^2\theta + r^2\sin^2\theta\right)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} r^2 d\theta$$

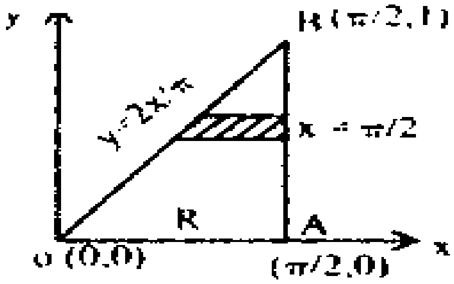
$$= \frac{1}{2}r^2[\theta]_0^{2\pi} = \pi r^2$$

Example 3:

Evaluate $\int_c [(\sin x - y)dx - \cos x dy]$ **where c is the triangle with vertices (0,0) ,(**$\frac{\pi}{2}$,0) **and** ($\frac{\pi}{2}$,1)

Solution: Equation of OB is $\frac{y-0}{1-0} = \frac{x-0}{\frac{\pi}{2}-0}$

$$\Rightarrow y = \frac{2x}{\pi}$$



By Green's theorem $\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$

Here $M = \sin x - y, \frac{\partial M}{\partial y} = -1$

N $= -\cos x, \frac{\partial N}{\partial x} = \sin x$

$\therefore \int_C [(\sin x - y)dx - \cos x dy] = \iint_R (\sin x + 1) dxdy$

In the region R, x varies from x = $\frac{\pi y}{2} \text{ to } \frac{\pi}{2}$ **and y varies from y = 0 to y = 1**

$$\therefore \int_C (\sin x - y)dx - \cos x dy = \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (\sin x + 1) dxdy$$

$$= \int_0^1 [-\cos x + x]_{\frac{\pi y}{2}}^{\frac{\pi}{2}} \text{ dy}$$

$$= \int_0^1 \left[\cos\frac{\pi y}{2} + \frac{\pi}{2} - \frac{\pi y}{2} \right] dy$$

$$= \left[\frac{2}{\pi}\sin\frac{\pi y}{2} + \frac{\pi}{2}y - \frac{\pi y^2}{4} \right]_0^1$$

$$= \frac{2}{\pi} + \frac{\pi}{2} - \frac{\pi}{4} = \frac{2}{\pi} + \frac{\pi}{2}$$

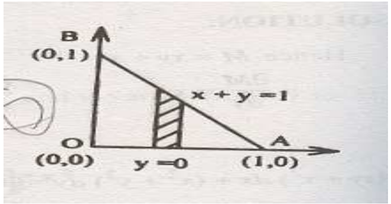
Example 4

Verify Green's theorem in the plane for
$\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ **where C is the boundary of the region defined by**
X = 0 , y= 0, x + y =1
Solution: We have to prove that

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$\mathbf{M} = 3x^2 - 8y^2, N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$$



By Green's theorem in the plane

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^{1-x} (10y) dydx$$

$$= 10\int_0^1 \left[\frac{y^2}{2} \right]_0^{1-x}$$

$$= 5\int_0^1 (1-x)^2 dx$$

$$= 5\left[\frac{(1-x)^3}{-3} \right]_0^1 = \frac{5}{3}$$

Consider $\int_c Mdx + Ndy = \int_{OA} + \int_{AB} + \int_{BO}$

Along OA, y=0 , x varies from 0 to 1

$$\therefore \int_{OA} Mdx + Ndy = \int_0^1 3x^2 dx = \left[x^3 \right]_0^1 = 1$$

Along AB, y = 1 - x $\Rightarrow dy = -dx$ **and x varies from 1 to 0 .**

$$\therefore \int_{AB} Mdx + Ndy = \int_1^0 \left[3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x) \right] dx$$

$$= \left[\frac{3x^2}{3} - \frac{8(1-x)^3}{-3} - \frac{4(1-x)^2}{-2} + 3x^2 - 2x^3 \right]_1^0$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = \frac{8}{3}$$

STOKE'S THEOREM

If S is an open surface bounded by a simple closed curve C and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in S and on C, then $\iint curl\,\vec{F}.\vec{n}\,ds = \int_c \vec{F}.d\vec{r}$ where $\vec{n}$ is the unit vector normal to the surface (ie) The surface integral of the normal component of $curl\,\vec{F}$ is equal to the integral of the tangential component of $\vec{F}$ taken around C.

Example 1

Verify Stoke's theorem for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the upper half of the sphere $x^2+y^2+z^2=1$ and C is the circular boundary on z = 0 plane.

Solution: By Stoke's theorem $\int_c \vec{F}.d\vec{r} = \iint_s curl\,\vec{F}.\vec{n}\,ds$

$$\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$$

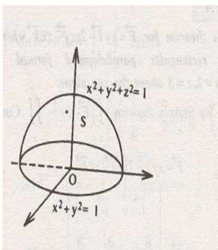
$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{i}[-2yz+2yz] - \vec{j}(0-0) + \vec{k}(0+1) = \vec{k}$$

Here $\vec{n} = \vec{k}$ **since C is the circular boundary on z = 0 plane**

$\therefore \iint_S$ = **area of the circle** $curl\,\vec{F}.\vec{n}\,ds = \iint_S dxdy$

$= \pi(1)^2 = \pi$ **.........(1)**



ON z = 0, $\int_c \vec{F}.d\vec{r} = \iint_s curl\,\vec{F}.\vec{n}\,ds$

On C, x = cos$\theta, y = \sin\theta$

$dx = -\sin\theta d\theta, dy = \cos\theta d\theta$

θ **varies from 0 to** 2π

$$\therefore \int_c \vec{F}.d\vec{r} = \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)d\theta$$

$$= -\int_0^{2\pi} (2\cos\theta\sin\theta)d\theta + \int_0^{2\pi} \sin^2\theta d\theta$$

$$= -\int_0^{2\pi} (\sin 2\theta)d\theta + \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= -\left[\frac{\cos 2\theta}{2}\right]_0^{2\pi} + \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= -\frac{1}{2} + \frac{1}{2} + \pi = \pi \quad \text{...............(2)}$$

$\therefore$ **From (1) and (2)**

$$\int_c \vec{F}.d\vec{r} = \iint_s curl\,\vec{F}.\vec{n}\,ds$$

Hence stoke's theorem is verified

Example 2

Verify stoke's theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$ **where s is the surface of the cube x = 0, x = 2, y = 0, y = 2, z = 0 and z = 2 above the xy – plane.**

Solution:

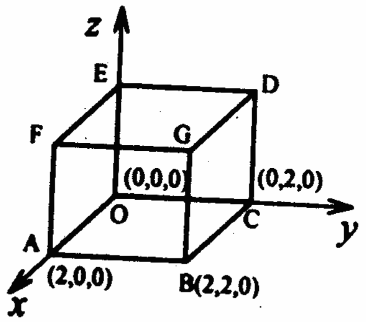
By Stoke's theorem

$$\int_c \vec{F}.d\vec{r} = \iint_s curl\,\vec{F}.\vec{n}\,ds$$

Given $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$ $\quad curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$

$$= \vec{i}[0 - y] - \vec{j}[-z + 1] + \vec{k}[0 - 1]$$

$$= -y\vec{i} + \vec{j}[z - 1] - \vec{k}$$



Surface	Equation	$\vec{n}$	ds	Curl $\vec{F}\cdot\vec{n}$
$ABEF\ (s_1)$	$x = 2$	$\vec{i}$	$dy\,dz$	$-y$
$OCDG\ (s_2)$	$x = 0$	$-\vec{i}$	$dy\,dz$	y
$BCDE\ (s_3)$	$y = 2$	$\vec{j}$	$dx\,dz$	
$OAFG\ (s_4)$	$y = 0$	$-\vec{j}$	$dx\,dz$	$-(z-1)$
$DEFG\ (s_5)$	$z = 2$	$\vec{k}$	$dx\,dy$	-1

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds = \iint_{s_1} + \iint_{s_2} + \iint_{s_3} + \iint_{s_4} + \iint_{s_5}$$

From curl $\vec{F}\cdot\vec{n}$ values all the first four integrals vanishes.

$$\therefore \iint_{s_5} \text{curl}\,\vec{F}\cdot\vec{n}\,dx\,dy = -\int_0^2\int_0^2 dx\,dy = -4$$

$$\therefore \iint_s \text{curl}\,\vec{F}\cdot\vec{n}\,ds = -4 \qquad \dots (1)$$

The boundary curve c is the face $OABC$, which is a square with sides, OA, AB, BC and CO.

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = (y - z + 2)\,dx + (yz + 4)\,dy - xz\,dz$$

$$= (y + 2)\,dx + 4dy \qquad [\text{since } z = 0]$$

Along OA, $y = 0$, $dy = 0$, x varies from 0 to 2.

$$\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_0^2 2dx = 4$$

Along AB, $x = 2$, $dx = 0$, y varies from 0 to 2

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^2 4\,dy = 8$$

Along BC, $y = 2$, $dy = 0$, x varies from 2 to 0

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_2^0 4\,dx = -8$$

Along CO, $x = 0$, $dx = 0$, y varies from 2 to 0

$$\int_{CO} \vec{F}\cdot d\vec{r} = \int_2^0 4dy = -8$$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = 4 + 8 - 8 - 8 = -4 \qquad \dots (2)$$

From (1) and (2)

Hence $$\int_c \vec{F}\cdot d\vec{r} = \int_s \int \text{curl}\,\vec{F}\cdot\vec{n}\,ds$$

Hence Stoke's theorem is verified.

Example 3:

Verify Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution: By stoke's theorem

$$\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds = \int_c \vec{F}\cdot d\vec{r}$$

$$\vec{F} = y\,\vec{i} + z\,\vec{j} + x\,\vec{k}$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \vec{i}[0-1] - \vec{j}[1-0] + \vec{k}[0-1] = -\vec{i} - \vec{j} - \vec{k}$$

Now $\phi = x^2 + y^2 + z^2 - 1$

$$\nabla\phi = 2x\,\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2$$

$$\therefore\ \vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\therefore\ \text{curl}\,\vec{F}\cdot\vec{n} = -x - y - z$$

S be the projection on XOY plane gives

$$ds = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \frac{dx\,dy}{z}$$

$$\therefore\ \iint_S (-x-y-z)\,\frac{dx\,dy}{z} = \iint_S \frac{(-x-y)}{z}\,dx\,dy - \iint_S dx\,dy$$

$$= \iint_S \frac{(-x-y)}{\sqrt{1-x^2-y^2}} - dx\,dy - \iint_S dx\,dy$$

$$= I_1 - I_2$$

Now $I_1 = \displaystyle\iint_S \frac{-x-y}{\sqrt{1-x^2-y^2}}\,dx\,dy$

By changing to polar coordinates

$x = r\cos\theta,\ y = r\sin\theta\quad dx\,dy = r\,dr\,d\theta$

r varies from 0 to 1

θ varies from 0 to 2π

$$I_1 = \int_0^{2\pi}\int_0^1 \frac{\cos\theta + \sin\theta}{\sqrt{1-r^2}}\, r^2\, dr\, d\theta$$

$$= -\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr \int_0^{2\pi} (\cos\theta + \sin\theta)\, d\theta$$

$$= -\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr\, [\sin\theta - \cos\theta]_0^{2\pi} = -\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr\, [-1+1] = 0$$

Now $I_2 = \iint_S dx\, dy$

$= \text{Area of the closed circle } x^2 + y^2 = 1 = \pi$

$\therefore \; I_1 - I_2 = 0 - \pi = -\pi$

$\therefore \; \iint_S curl\,\vec{F} \cdot \vec{n}\, ds = -\pi$ ...(1)

$\vec{F} \cdot d\vec{r} = ydx + zdy + xdz$

The equation of the boundary circle is $x^2 + y^2 = 1$

since $z = 0, dz = 0$, we have $\vec{F} \cdot d\vec{r} = ydx$

put $x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

θ varies from 0 to 2π

$$\int_c \vec{F} \cdot d\vec{r} = -\int_0^{2\pi} \sin\theta^2\theta\, d\theta = -\frac{1}{2}\int_0^{2\pi} (1 - \cos 2\theta)\, d\theta$$

$$= -\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= -\pi$$ ... (2)

$\therefore$ From (1) and (2)

$$\int_c \vec{F}.d\vec{r} = \iint_s curl\,\vec{F}.\vec{n}\,ds$$

Gauss Divergence theorem

Statement:

The surface integral of the normal component of a vector function F over a closed surface S enclosing volume V is equal to the volume integral of the divergence of F taken throughout the volume V ,

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}dv$$

Evaluate $\iint x^3 dydz + x^2 ydzdx + x^2 zdxdy$ **over the surface bounded by z = 0 ,z = h,** $x^2 + y^2 = a^2$

Solution:

$$\iint_s (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_v \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

Here $F_1 = x^3,\ F_2 = x^2y,\ F_3 = x^2z$

$$\therefore \frac{\partial F_1}{\partial x} = 3x^2,\quad \frac{\partial F_2}{\partial y} = x^2,\quad \frac{\partial F_2}{\partial z} = x^2$$

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 5x^2$$

$$\therefore \iint_s F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_v 5x^2\,dx\,dy\,dz$$

$$= \int_{z=0}^{h} \int_{y=-a}^{a} \int_{x=\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 5x^2\,dx\,dy\,dz$$

$$= \int_0^h \int_{-a}^{a} \left[\frac{10\,x^3}{3}\right]_0^{\sqrt{a^2-y^2}} dy\,dz$$

$$= \int_0^h \int_{-a}^{a} \frac{10}{3}(a^2-y^2)^{3/2}\,dy\,dz$$

$$= \frac{20}{3} \int_0^h \int_0^a (a^2-y^2)^{3/2}\,dy\,dz$$

Put $x = a\cos\theta\ y = a\sin\theta$

$dy = a\cos\theta\,d\theta$

$\Rightarrow \theta$ varies from 0 to $\pi/2$

$$\therefore \iint_s F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy$$

$$= \frac{20}{3} \int_0^h \int_0^{\pi/2} (a^2 - a^2\sin^2\theta)^{3/2}\,a\cos\theta\,d\theta\,dz$$

$$= \frac{20a^4}{3} \int_0^h \int_0^{\pi/2} \cos^4\theta\,d\theta\,dz$$

$$= \frac{20a^4}{3} \int_0^h \frac{3\pi}{16}\, dz = \frac{5}{4} a^4 \pi \int_0^h dz$$

$$= \frac{5a^4\pi}{4} h$$

$$\int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta = \frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{3\pi}{16}$$

Evaluate $\iint_s \vec{F} \cdot \vec{n}\, ds$ ***where*** $\vec{F} = 4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k}$ **and** s ***is the surface of the cube bounded by*** $x = 0, x = a, y = 0, y = a, z = 0, z = a$ **[$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ means $a = 1$]**

☞ **SOLUTION:** Given $\vec{F} = 4xz\,\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$

$$\text{div}\,\vec{F} = 4z - 2y + y = 4z - y$$

x ; varies from 0 to a

y ; varies from 0 to a

z ; varies from 0 to a

$$\iint_s \vec{F} \cdot \vec{n}\, ds = \iiint_v \text{div}\,\vec{F}\, dv$$

$$= \int_0^a \int_0^a \int_0^a (4z - y)\, dx\, dy\, dz$$

$$= \int_0^a \int_0^a [4zx - yx]_0^a\, dy\, dz$$

$$= \int_0^a \int_0^a [4za - ya]\, dy\, dz$$

$$= \int_0^a \left[4azy - \frac{ay^2}{2}\right]_0^a dz = \int_0^a \left(4a^2 z - \frac{a^3}{2}\right) dz$$

$$= \left[2a^2 z^2 - \frac{a^3 z}{2}\right]_0^a = 2a^4 - \frac{a^4}{2}$$

$$\iint_S \vec{F} . \vec{n}\, ds = \frac{3a^4}{2}$$